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Forced Random Parametric Vibration in Single-Degree-of-Freedom Systems

D. E. Huntington* and C. S. Lyrintzis†
San Diego State University, San Diego, California 92182-0183

Linear systems with random properties subjected to stochastic loading are difficult to analyze by conventional means, and most analysts simply ignore the random property contribution to the system response. The relative importance of the random property effects in the system response under a variety of conditions is examined. Toward this end, a random matrix technique is proposed that is more flexible and more computationally efficient than conventional approaches when dealing with this class of problem. This technique, which is an improved variation of the perturbation approach, will be used to analyze single-degree-of-freedom linear oscillators with various damping ratios and property variation types, subjected to stationary loading. Some of these cases will also be analyzed by Monte Carlo simulation, for comparison. Finally, trends in the importance of random property effects will be discussed.

Introduction

THERE are two main sources of randomness in the response of a vibrating system: randomness in the external loading and randomness in the system properties. Wind, ocean waves, and earthquakes all give rise to random loading on a structure. Some problems of interest have effective properties that are random in nature, such as helicopter rotor blade vibration in turbulent flow[1] or ship roll dynamics.[2] Further, uncertainties in the properties of any structure can arise from manufacturing or construction processes and from randomness in live load. Systems that have time-varying properties and no external forcing are said to undergo parametric vibration. In this work, the term forced random parametric vibration (FRPV) will refer to any system with both random properties (which may or may not vary in time) and dynamic random external loading. From the preceding examples, it is seen that FRPV occurs fairly often in practice.

However, it is relatively difficult and computationally expensive to analyze FRPV by conventional means; it is far simpler to analyze a linear system with deterministic properties subjected to stochastic loading. This work investigates the magnitude of the random property effects in the response of single-degree-of-freedom (SDOF) systems and attempts to discover under what conditions the simpler analysis may be used for an FRPV problem. Thus, a variety of FRPV systems must be analyzed by a reliable, flexible, accurate, easy to use, and fast computational technique. But what technique should be used?

A good overview of early research in forced and unforced random parametric vibration (especially in SDOF systems) is found in a monograph written by Ibrahim.[3] Much of this early research analyzed random parametric vibration using Markov methods based on Itô stochastic calculus or the Fokker–Planck–Kolmogorov equation; differential equations in the random moments of the response (the means, correlations, and so on) can be obtained from these methods. However, this approach can only be used for systems that have loading and properties that are Gaussian noise or filtered noise processes; the general loading process used in this work does not fit this criterion. Also, the resulting random moment equations often come in an infinite hierarchy, where first-order moments depend on second-order ones, and so on. Closure schemes have been developed to meet this problem, but they are often difficult to implement and are arbitrary in usage.

Another approach mentioned in Ref. 3 is stochastic averaging, which assumes that the response process will be nearly harmonic, with an amplitude and phase angle that varies slowly in time. This approach works well for a system with small variations in stiffness and small external loading, but many problems of interest (including those studied here) do not satisfy these conditions, and even when stochastic averaging becomes feasible, the resulting equations

In Memoriam

On Thursday, August 15, 1996, Dr. Constantinos (Costas) Lyrintzis, faculty member in the Department of Aerospace Engineering and Engineering Mechanics at San Diego State University (SDSU), was fatally shot while participating in an M.S. thesis defense. Dr. Lyrintzis, a native of Greece, was born September 22, 1960. He received a Diploma in Civil Engineering from the National Technical University of Athens in 1983 and M.S. and Ph.D. degrees in engineering mechanics from Columbia University in 1984 and 1987, respectively. He then joined the faculty at SDSU, where he was an associate professor at the time of his death.

Dr. Lyrintzis had been an active member of AIAA since 1987. He was an outstanding and caring teacher and mentor to undergraduate and graduate students. He developed a strong research program in structural dynamics, applying analytical and semianalytical techniques to study problems in vibrations and acoustics. Several of the articles describing this research have appeared in the *AIAA Journal* and the *Journal of Aircraft*.

Dr. Lyrintzis is survived by his wife Deana, his daughter Sofia, his brother Anastasios (a faculty member in the School of Aeronautics and Astronautics of Purdue University and an AIAA member), and his parents Angeliki and Sotirios Lyrintzis in Greece.

Costas Lyrintzis was a gentle young man, always with a smile, a friend to students and colleagues. His presence will be sorely missed.

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*Research Associate, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.
†Associate Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.

become quite complicated and difficult to solve. Thus, Markov methods and stochastic averaging approaches are useful for some problems in random parametric vibration, but they are not suitable for the analysis of a more general problem that is examined here.

Other computational techniques used to analyze systems with random properties subject to deterministic loading could perhaps be adapted to analyze FRPV. These techniques are currently used in conjunction with stochastic finite elements. One such technique recently proposed by Ghanem and Spanos[4] centers on a Karhunen–Loeve expansion involving the eigenfunctions of the property covariance function. These eigenfunctions are relatively easy to obtain for noise or filtered-noise properties but not for more general property variation. Further, it would be difficult to extend this approach to systems with time-varying properties or to systems subject to stochastic loading. Other researchers who have contributed to stochastic finite element approaches include Der Kiureghian,[5] Shinozuka and Deodatis,[6] and Schuëller and Bucher,[7] among others; however, their techniques are also difficult to extend to systems with time-varying random properties.

On the other hand, a natural candidate for a general solution technique for FRPV is Monte Carlo simulation. The method generates sample realizations of the random fields or processes in the problem, solves the problem deterministically for each realization, and finally uses the sample results to obtain statistics of the response. The sample generation scheme is the backbone of this method, and several generation schemes have been used, such as the autoregressive moving average technique,[8,9] the spectral representation approach,[10,11] and Latin hypercube sampling.[12] Monte Carlo simulation has been used to solve a variety of random response problems, including stochastic finite element problems (for example, Ref. 13). Note that Monte Carlo sampling schemes can only match prescribed means and correlations; higher-order information cannot be used. In practice, Monte Carlo schemes are robust and relatively simple to use but require many samples, and thus much computation time, for accurate results.

Monte Carlo simulation has been used by Seya et al.[14] in a recent analysis of a five-story steel frame structure with uncertain properties subjected to an earthquake, which is an example of FRPV. They used Latin hypercube sampling to generate realizations of ground motions, member yield strength (used for bilinear hysteretic stiffness), and system damping, and used a two-dimensional nonlinear solver for the deterministic seismic analysis. They examined structural fragility and related the structure's response modification factor to the ductility ratio. However, their modeling of the damping ratio was simplistic (assuming that the damping ratio was 1, 2, or 3% with equal probability), and they used 18 sample systems for their analysis, which is far too few. Nonetheless, the potential for using Monte Carlo simulation to analyze a difficult problem in FRPV was shown.

A computational technique that may also be suitable for FPRV analysis is the perturbation approach. The desired response is expanded in a Taylor series in the system's fundamental random variables, which govern the system's properties and loading. The statistics of these fundamental random variables are used to obtain the response statistics. The perturbation technique has been employed in conjunction with stochastic finite element concepts by Hisada and Nakagiri to analyze geometric randomness in fracture analysis[15] and to solve a variety of static and dynamic problems.[16] The perturbation technique has also been recently used by Cai and Lin[17] to examine localization effects in disordered periodic structures. However, because of stochastic convergence problems in the Taylor expansion, the perturbation approach is limited to small property variation. Also, if higher-order information is used, or if the fundamental random variables are complicated, the perturbation approach becomes computationally intensive.

This work will introduce a random matrix method, which is an extension of the perturbation method, to analyze SDOF systems undergoing FRPV, with stationary properties and loading. The random matrix method takes advantage of property and load stationarity, when it exists; it also significantly reduces required computation time and storage requirements compared with the regular perturbation approach. This method will be used to analyze SDOF systems with different damping ratios, magnitudes of property variation, and property correlation lengths, to find the relative importance of the property effects on the system's response. Where feasible, results obtained by the random matrix method will be compared with those obtained by direct probability integration or Monte Carlo simulation, to compare results from the proposed method. Limitations of the random matrix method will also be addressed. Finally, trends in the relative importance of the random properties will be discussed.

Random Matrix Method

Before developing the random matrix method, it is helpful to examine a standard perturbation analysis of a multiple-degree-of-freedom system under deterministic static load (which may result from a stochastic finite element analysis). The perturbation method expands the random stiffness matrix $[\boldsymbol{K}]$ in a Taylor series in the fundamental random property variables $a_1, \ldots, a_n$, as follows[15]:

$$\boldsymbol{K} = \boldsymbol{K}^0 + \sum_{i=1}^{n} \boldsymbol{K}_i^1 a_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{K}_{ij}^{11} a_i a_j + \cdots \tag{1}$$

where $\boldsymbol{K}^0$, $\boldsymbol{K}^1$, and $\boldsymbol{K}^{11}$ are coefficient matrices. A similar Taylor expansion is developed for the deflection vector. Statistics of the deflection vector are therefore obtained from the statistics of the a_i.

If the stiffness variation is moderate or large, or if the stiffness matrix depends on the fundamental random variables in a complicated way, then the stiffness matrix should be expanded to second order or higher. For n fundamental random variables, a second-order stiffness matrix expansion would require the storage of n^2 coefficient stiffness matrices, each of which is $d \times d$ elements in size for d degrees of freedom. Also, the computation of response statistics for a second-order expansion would require roughly $n^4 d^2$ multiplications. Thus, second-order perturbation solutions require too much computer storage space and computation time to be practical.

Another difficulty with the standard perturbation approach, as mentioned previously, is the question of whether the deflection's Taylor expansion will converge. If the property variations are within reasonable bounds (to prevent nonpositive stiffness, for example), then the Taylor expansion must converge for static problems, though if the property variations are large, second-order or higher expansions may be needed to compute response statistics with reasonable accuracy, requiring much computation time and memory. On the other hand, dynamic problems with moderate-to-large property variations and small damping ratios yield divergent Taylor expansions for the deflection. This problem is more serious than the last because it limits the type of problem that the perturbation approach can analyze, no matter how much computation time or storage space is available. These problems with storage requirements, computation time, and divergent Taylor expansions must be addressed by any variant or extension of the perturbation approach, including the proposed random matrix method.

To see how the random matrix method works, and to address the problems just mentioned, a SDOF system undergoing FRPV will be examined.[18] The differential equation of this oscillator is

$$m(t)\ddot{x}(t) + c(t)\dot{x}(t) + k(t)x(t) = f(t) \tag{2}$$

where the mass, damping, and stiffness, $m(t)$, $c(t)$, and $k(t)$, respectively, are stationary, Gaussian random processes that depend on the time t, and the loading $f(t)$ can be any random process. The properties are assumed to be independent of the loading and of each other, for purposes of simplicity in this analysis, but the random matrix technique is not limited to this case.

To solve this problem in frequency space, the Fourier transform (FT) of both sides of Eq. (2) is needed. The first term on this side is $m(t)\ddot{x}(t)$. Because $m(t)$ is stationary and real, it has a mean $\bar{m}$ that is independent of time. The deviatoric part of the mass, $m'(t) = m(t) - \bar{m}$, has an FT $\tilde{m}(\omega)$ with the following properties:

$$\begin{gathered} E[\tilde{m}(\omega)] = 0 \\ E[\tilde{m}_R(\omega)\tilde{m}_R(\omega^*)] = \pi S_m(\omega)[\delta(\omega - \omega^*) + \delta(\omega + \omega^*)] \\ E[\tilde{m}_I(\omega)\tilde{m}_I(\omega^*)] = \pi S_m(\omega)[\delta(\omega - \omega^*) - \delta(\omega + \omega^*)] \\ E[\tilde{m}_R(\omega)\tilde{m}_I(\omega^*)] = 0 \end{gathered} \tag{3}$$

where $E[\cdot]$ denotes expected value, $S_m(\omega)$ is the spectral density of the deviatoric mass, ω and ω^* denote frequencies, δ is the Dirac delta function, and the subscripts R and I denote real and imaginary parts, respectively. Therefore, the FT of $m(t)\ddot{x}(t)$, denoted by $\widetilde{m\ddot{x}}(\omega)$, is

$$\widetilde{m\ddot{x}}(\omega) = -\omega^2 \tilde{m}\tilde{x}(\omega) + \frac{1}{2\pi}\int_{-\infty}^{\infty} -\omega_0^2 \tilde{m}(\omega-\omega_0)\tilde{x}(\omega_0)\,d\omega_0 \quad (4)$$

The FTs of the other terms on the left-hand side of Eq. (2) are obtained in a similar fashion. Once this is done, a relation between the FT of the deflection $\tilde{x}$ and the FT of the load $\tilde{f}$ may be developed:

$$\tilde{f}(\omega) = [-\omega^2\tilde{m} - \hat{\imath}\omega\tilde{c} + \tilde{k}]\tilde{x}(\omega) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\big[-\omega_0^2\tilde{m}(\omega-\omega_0) - \hat{\imath}\omega_0\tilde{c}(\omega-\omega_0) + \tilde{k}(\omega-\omega_0)\big]\tilde{x}(\omega_0)\,d\omega_0 \quad (5)$$

where $\hat{\imath} = \sqrt{(-1)}$.

Solving the previous equation for $\tilde{x}$ could be quite difficult. A simpler approach is to discretize the frequency domain, so that the previous equation reduces to

$$[\boldsymbol{A}]\{\tilde{x}\} = \{\tilde{f}\} \quad (6)$$

where

$$\boldsymbol{A}_{ij} = \frac{1}{\sqrt{2}}\big[-\omega_j^2(\tilde{m}_{i-j} + \tilde{m}_{j-i}) + \hat{\imath}\omega_j(\tilde{c}_{i-j} + \tilde{c}_{j-i}) + (\tilde{k}_{i-j} + \tilde{k}_{j-i}) + \big(-\omega_j^2\bar{m} + \hat{\imath}\omega_j\bar{c} + \bar{k}\big)\delta_{ij}\big] \quad (7)$$

where the overbar denotes the mean of a quantity, and δ_{ij} is the Kronecker delta. Because of the discretization, $\tilde{m}$, $\tilde{c}$, and $\tilde{k}$ are vectors of length $2N+1$ when N positive discrete frequencies are used, as well as the corresponding negative frequencies and zero frequency. The indices i and j range from $-N$ to N, and so do their differences (if the differences are outside of the range, multiples of $2N+1$ are added or subtracted as needed).

At this point, the matrix $[\boldsymbol{A}]$ is separated into a mean $[\bar{\boldsymbol{A}}]$ and deviation $[\boldsymbol{A}']$:

$$[\boldsymbol{A}] = [\bar{\boldsymbol{A}}] + [\boldsymbol{A}'] \quad (8)$$

where $[\bar{\boldsymbol{A}}]$ is a diagonal matrix given by

$$\bar{\boldsymbol{A}}_{ij} = \big(-\omega_j^2\bar{m} + \hat{\imath}\omega_j\bar{c} + \bar{k}\big)\delta_{ij} \quad (9)$$

To invert the matrix $[\boldsymbol{A}]$, the Neumann expansion[6,13] is used:

$$[\boldsymbol{A}]^{-1} = (\boldsymbol{I} - \boldsymbol{P} + \boldsymbol{P}^2 - \boldsymbol{P}^3 + \cdots)[\bar{\boldsymbol{A}}]^{-1} \quad (10)$$

where the matrix $[\boldsymbol{P}]$ is

$$[\boldsymbol{P}] = [\bar{\boldsymbol{A}}]^{-1}[\boldsymbol{A}'] \quad (11)$$

The Neumann expansion in Eq. (10) is identical to the Taylor expansion used in the perturbation technique if the fundamental random variables in the perturbation approach are taken to be the real and imaginary parts of the stiffness matrix elements. Since the Neumann expansion does not store coefficient matrices, but directly computes correlations between elements in powers of the matrix $[\boldsymbol{P}]$, the random matrix method requires less storage space than the standard perturbation method does.

Another difference between the standard perturbation method and the random matrix approach should be mentioned. A perturbation approach first truncates the deflection expansion and then finds the deflection spectral density based on the truncated deflection. On the other hand, the random matrix approach finds the full series expansion for the spectral density directly and then truncates it. To clarify this point, consider the relationship between deflection, loading, and properties in the frequency domain, shown in Eq. (6). If the loading and properties are independent, then the correlations between elements in $\{\tilde{x}\}$ (which are used to obtain deflection spectral density) can be found from the correlations between elements in $[\boldsymbol{A}]^{-1}$ and in $\{\tilde{f}\}$. Now consider the expansion for $[\boldsymbol{A}]^{-1}$, as shown in Eq. (10). The correlation between two elements in $[\boldsymbol{A}]^{-1}$ could also be represented in a series expansion. Now, suppose that the series for a property correlation is to be expanded to second order. Appropriate elements in $[\boldsymbol{P}]$ would be correlated with each other, and means of elements in $[\boldsymbol{P}]^2$ would also be used (effectively correlating elements in $[\boldsymbol{P}]^2$ with elements in the identity matrix $[\boldsymbol{I}]$). This is the approach that the random matrix method takes. On the other hand, if the expansion in Eq. (10) is truncated to first order, and then elements in the truncated expansion were correlated, only correlations in $[\boldsymbol{P}]$ would be used. This is the approach taken by the perturbation solution, and it leaves out important second-order terms as a result. Thus, the random matrix approach uses more terms of the same order for spectral density than an equivalent perturbation solution, making the random matrix results more accurate.

The real and imaginary part of each element in matrix $[\boldsymbol{P}]$ is Gaussian with mean zero if the properties are Gaussian and stationary. It is straightforward to determine the covariances between the real or imaginary part of any two elements in $[\boldsymbol{P}]$, which can be used to determine covariances between real and imaginary parts of elements in $[\boldsymbol{P}]^n$ (because elements in $[\boldsymbol{P}]$ are Gaussian). The means and covariances of elements in the truncated Neumann expansion of $[\boldsymbol{A}]^{-1}$ can be computed from the covariances in $[\boldsymbol{P}]$, and if higher-order random moments of $[\boldsymbol{A}]^{-1}$ are needed, they can also be directly computed. (Note that the means and covariances are random moments of orders one and two, respectively.) Further, the covariances between P_{ij} and P_{kl} are zero unless $(i-j) = \pm(k-l) + n(2N+1)$, where n is any integer. The random matrix method takes advantage of this, saving on computation time.

The analysis of the problem is further simplified if the loading is stationary, since the deflection would also be stationary. The mathematical properties that applied to $\tilde{m}(\omega)$ in Eq. (3) would also apply to $\tilde{f}(\omega)$ and $\tilde{x}(\omega)$. A result of this is that a direct relationship between the deflection's spectral density $S_x(\omega)$ and the load's spectral density $S_f(\omega)$ can be derived:

$$S_x(\omega_i) = \sum_{j=-N}^{N} E\big[\big|A_{ij}^{-1}\big|^2\big]S_f(\omega_j) \quad (12)$$

Thus, only variances of each element in $[\boldsymbol{A}]^{-1}$ are needed to determine the desired results, speeding up the analysis considerably. The deflection's autocorrelation function can then be found from its spectral density by use of a discrete Fourier transform. For ease of analysis, all results in this work use stationary properties and stationary loading.

Although the random matrix method is faster and more accurate and requires less storage than the standard perturbation method, stochastic convergence problems should be investigated. To illustrate this point, consider an oscillator whose properties are random but do not vary in time; thus, $[\boldsymbol{A}]^{-1}$ would be diagonal. The Neumann expansion of $[\boldsymbol{A}]^{-1}$ would break down first at the element corresponding to the undamped natural frequency of the system, because the corresponding element in the diagonal $[\boldsymbol{A}]$ has the smallest mean value. It can be shown that the Neumann expansion diverges at that element when

$$\begin{aligned} COV_k &\approx \zeta \\ COV_m &\approx \zeta \\ COV_c &\approx 1 \end{aligned} \quad (13)$$

where ζ is the damping ratio of the system; subscripts k, m, and c refer to the stiffness, mass, and damping, respectively; and the coefficient of variation (COV) for a property is given by

$$COV_x = \sigma_x/\bar{x} \quad (14)$$

where σ_x is the standard deviation of property x. Thus, for lightly damped systems, it would seem that property variations should be restricted if the system is to be analyzed by the random matrix method. The precise effects of stochastic convergence problems on results obtained by the random matrix method will be investigated in the results section.

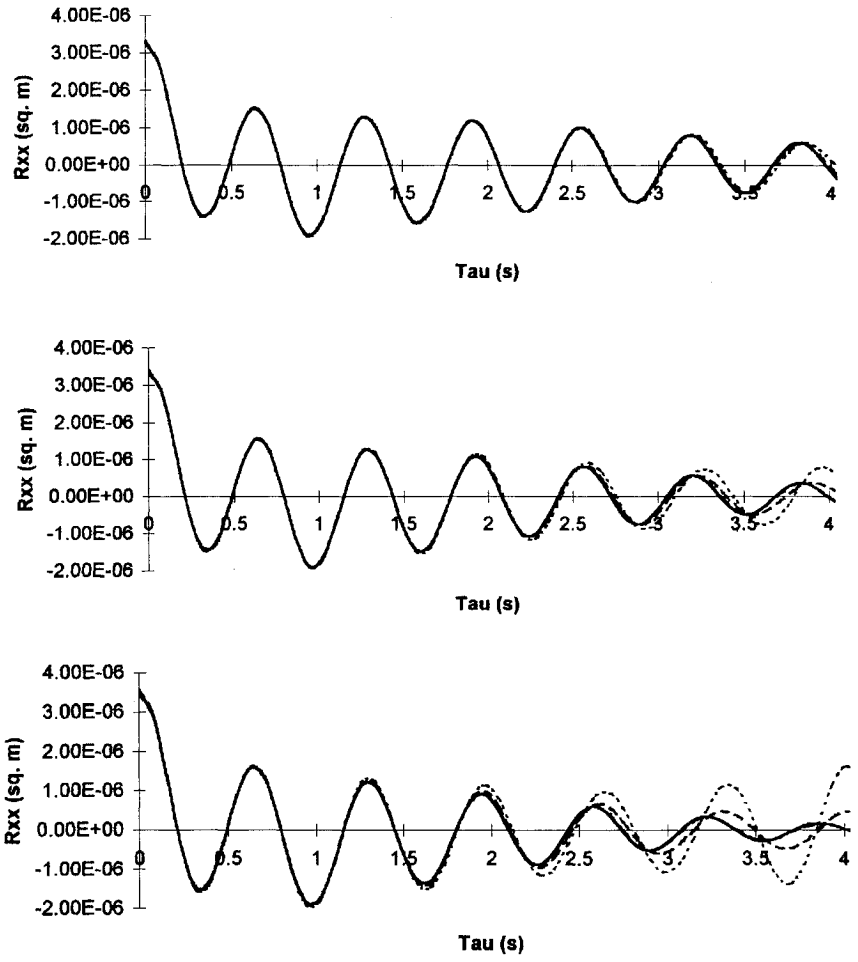

Fig. 1 Autocorrelation of deflection obtained by various techniques for a system with damping ratio $\zeta = 0.01$, stiffness correlation parameter $b = \infty$, and stiffness coefficients of variation 0.06, 0.08, and 0.10, respectively: ——, direct integration; - - - -, random matrix; – – – –, Monte Carlo; and – – –, Monte Carlo, Runge–Kutta.

Other Approaches for Comparison

To investigate the accuracy and stochastic convergence problems of the proposed method, some results obtained from the random matrix technique will be compared with those generated by other approaches. There are two approaches used to generate these comparison results: direct probability integration and Monte Carlo simulation.

Direct probability integration can be used when the random properties do not vary in time; in other words, the properties can be described by random variables instead of random processes. For simplicity, suppose that the mass m and damping c are deterministic, whereas the stiffness k is a Gaussian random variable truncated at five standard deviations. The resulting SDOF system has a second-order frequency transfer function $H(\omega)$ given by

$$H(\omega) = \int_{\bar{k}-5\sigma_k}^{\bar{k}+5\sigma_k} \frac{1}{k^2 + (c^2 - 2km)\omega^2 + m^2\omega^4} \times \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left[\frac{-(k-\bar{k})^2}{2\sigma_k^2}\right] dk \tag{15}$$

and the spectral density of the deflection can be found by

$$S_x(\omega) = H(\omega)S_f(\omega) \tag{16}$$

The integral in Eq. (15) is not solvable in closed form, but it can be solved numerically; fortunately, good numerical integration packages are readily available. The authors used the standard numerical integrator in Mathcad 5.0 Plus, the Romberg algorithm with trapezoidal integration approximation, to generate the desired results. However, this direct probability integration approach becomes infeasible when more than one property is random, or when a property varies in time.

As mentioned in the Introduction, Monte Carlo simulation is robust but computationally expensive for accurate results. Since Monte Carlo simulation does not suffer from stochastic convergence problems, the accuracy of Monte Carlo results is governed primarily by the number of samples taken, and so the accuracy of the random matrix method can be checked by comparing results obtained by the random matrix and Monte Carlo approaches. One Monte Carlo approach is used for systems with time-invariant random properties: the properties are simulated, and for each sample set of properties Eq. (7) is used to compute the property matrix and Eq. (12) is used to compute the deflection spectral density; finally, averages of the sample deflection spectral densities are computed. This direct frequency domain Monte Carlo approach is relatively fast but becomes impractical when the properties vary in time.

For time-varying property cases, the Monte Carlo approach utilizes a modification of Shinozuka's spectral representation sample generating scheme[10,11] to acquire sample functions for the time-varying stiffness and loading processes and solves for the deflection in the time domain by use of a fourth-order Runge–Kutta approach for systems of differential equations.[19] First, sample functions for the loading and properties were generated using this scheme:

$$a(t) = \sum_i r_i \cos(\omega_i t + \Phi_i) \tag{17}$$

where $a(t)$ represents the load or property being simulated, i ranges over the total number of nonnegative discrete frequencies involved,

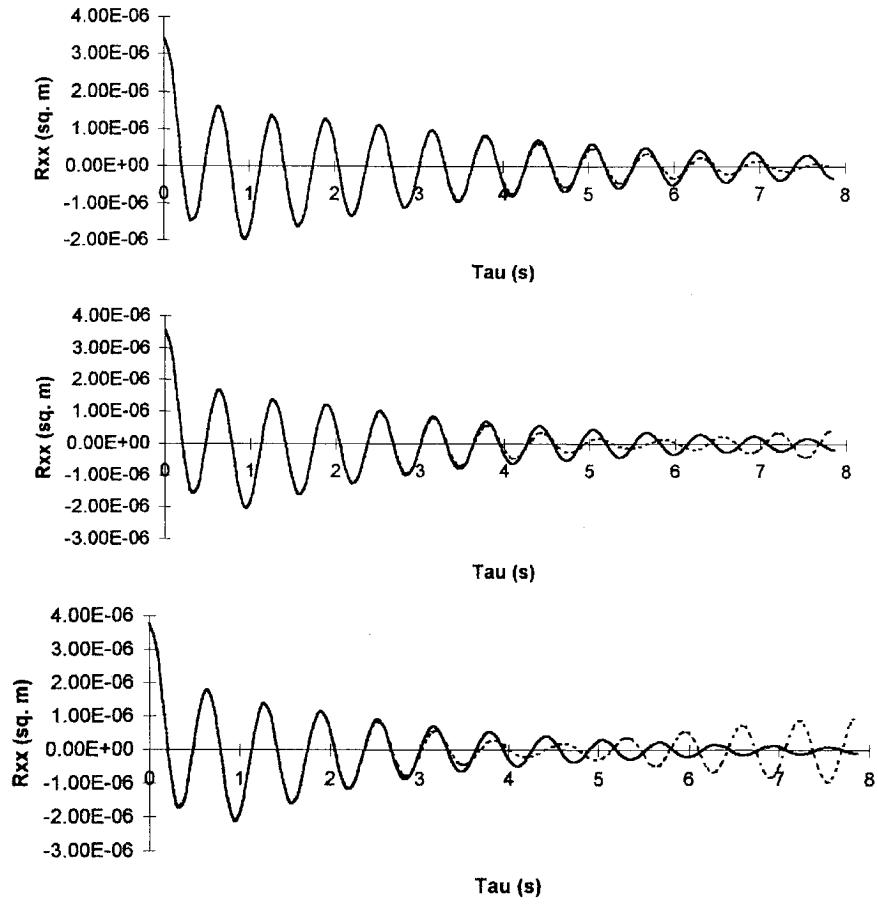

Fig. 2 Autocorrelation of deflection obtained by random matrix and Monte Carlo analysis for a system with damping ratio $\zeta = 0.01$, stiffness correlation parameter $b = 1$, and stiffness coefficients of variation 0.08, 0.10, and 0.12, respectively: ——, Monte Carlo and - - - -, random matrix.

Table 1 Order of magnitude estimates for storage space requirements of various first-order techniques with n total frequency steps

Method	$b_k = \infty$	b_k finite
Monte Carlo (Runge–Kutta)	n	n
Monte Carlo (frequency)	n^2	n^2
Perturbation	n^3	n^5
Random matrix	n	n

Table 2 Order of magnitude estimates for computer time requirements for various first-order techniques with n total frequency steps, using s samples for Monte Carlo analysis

Method	$b_k = \infty$	b_k finite
Monte Carlo (Runge–Kutta)	sn^2	sn^2
Monte Carlo (frequency)	sn	sn^3
Perturbation	n^4	n^5
Random matrix	n	n^2

the Φ_i are independent, uniformly distributed random variables over $[0, 2\pi)$, and the r_i are Rayleigh-distributed random variables such that

$$E[r_i^2] = 4S_a(\omega_i)\Delta\omega \tag{18}$$

For consistency, the spectral density at zero should be halved before applying this formula. These generated loading and property functions are evaluated at every half time step, because of the requirements of the Runge–Kutta solution technique; however, since these functions are periodic, they need only be evaluated over one period.

For usage in the standard Runge–Kutta approach, the single second-order differential equation of the system, Eq. (2) is transformed into two coupled first-order equations:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= [1/m(t)][f(t) - c(t)y(t) - k(t)x(t)] \end{aligned} \tag{19}$$

Deterministic initial conditions for the system were assumed: $x(0) = y(0) = 0$. The full time step size for the Runge–Kutta algorithm was set at half the step size used in random matrix analysis; this was necessary to ensure accuracy of the method. The Runge–Kutta algorithm was allowed to run for enough time steps (12,000) that the initial conditions did not have an appreciable effect on the results obtained for the last period (4002 time steps) of each function. The autocorrelation of deflection was computed by averaging over the last period of each sample deflection history and then averaging over the sample histories.

To illustrate the advantages that the random matrix approach has over existing techniques, rough order-of-magnitude analyses were done for each method for time-invariant properties ($b_k = \infty$) and time-varying properties (b_k finite). The storage space requirements are shown in Table 1, and the computer time requirements are shown in Table 2. The standard perturbation approach has the highest storage space and computer time requirements, whereas the random matrix approach has the smallest. Note that coefficients of these powers are not computed, but for many frequency steps it is clear which techniques are faster and require less storage space. Further, note that both Monte Carlo variants have roughly the same accuracy if the same number of samples is used.

Comparisons

This section contains results that illustrate the accuracy and limitations of the random matrix method. All results obtained in this

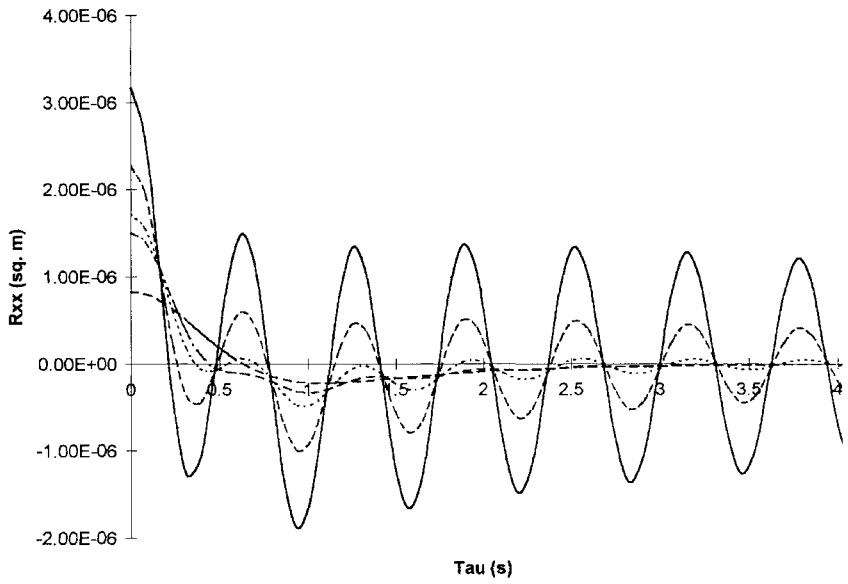

Fig. 3 Autocorrelation of deflection for deterministic stiffness with various damping ratios: ——, $\zeta = 0.01$; – – –, $\zeta = 0.02$; - - - -, $\zeta = 0.05$; – – –, $\zeta = 0.1$; and – - - -, $\zeta = 1$.

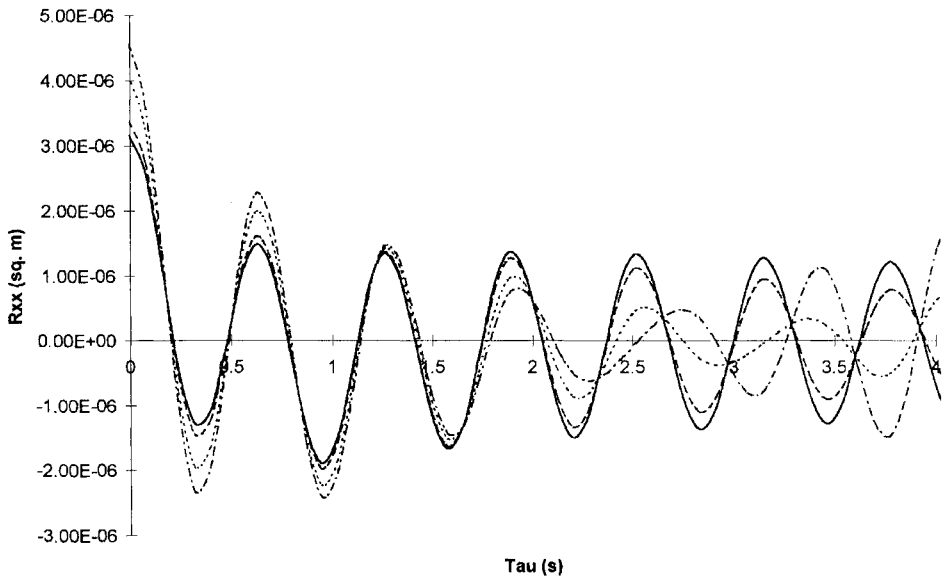

Fig. 4 Autocorrelation of deflection obtained by random matrix analysis for damping ratio $\zeta = 0.01$, stiffness correlation parameter $b = 1$, and various stiffness coefficients of variation: ——, $COV = 0$; – – –, $COV = 0.08$; - - - -, $COV = 0.16$; and – - –, $COV = 0.20$.

section refer to a SDOF system with random loading and random stiffness. The mass in all cases is 10 kg, and the mean stiffness in all cases is 1000 N/m, giving a mean undamped natural frequency of 10 rad/s. The loading in all cases has zero mean and a spectral density given by

$$S_f(\omega) = \tfrac{1}{4}\omega^2 e^{-|\omega|}\ \mathrm{N}^2/(\mathrm{rad/s}) \tag{20}$$

whereas the spectral density for the deviatoric part of the stiffness is

$$S_k(\omega) = \frac{\sigma_k^2}{\pi}\frac{b_k}{1 + b_k^2\omega^2} \tag{21}$$

where σ_k is the standard deviation of the stiffness and b_k is a parameter relating to the correlation time of the stiffness process. Note that if b_k approaches infinity, k becomes a random variable instead of a random process. In each case, 2001 total frequencies were used (1000 positive frequencies, their opposites, and zero), with a frequency step size of 0.04 rad/s, giving a standard time step of 0.0785 s. This information is summarized in Table 3. Note that different cases will have different damping ratios, stiffness coefficients of variation [related to σ_k through Eq. (14)], and stiffness correlation parameters.

Table 3 Values of various parameters used to compute results

Parameter	Symbol	Value[a]
Mass	m	10 kg
Damping	c	*
Stiffness mean	$\bar{k}$	1000 N/m
Stiffness standard deviation	σ_k	*
Stiffness correlation parameter	b_k	*
Frequency step size	$\Delta\omega$	0.04 rad/s
Time step size	Δt	0.0785 s

[a]Parameters with an asterisk (*) depend on the specific case examined, and the specific forms of the stiffness and load distribution are found in the text.

The first set of deflection autocorrelation results for comparison are shown in Fig. 1 for a mean damping ratio $\zeta = 0.01$ and a time-invariant stiffness distribution ($b_k = \infty$) with coefficient of variation [see Eq. (14)] $COV = 0.06$, 0.08, and 0.10. As indicated, results were obtained by direct integration, by the random matrix technique, by Monte Carlo simulation of the stiffness in the

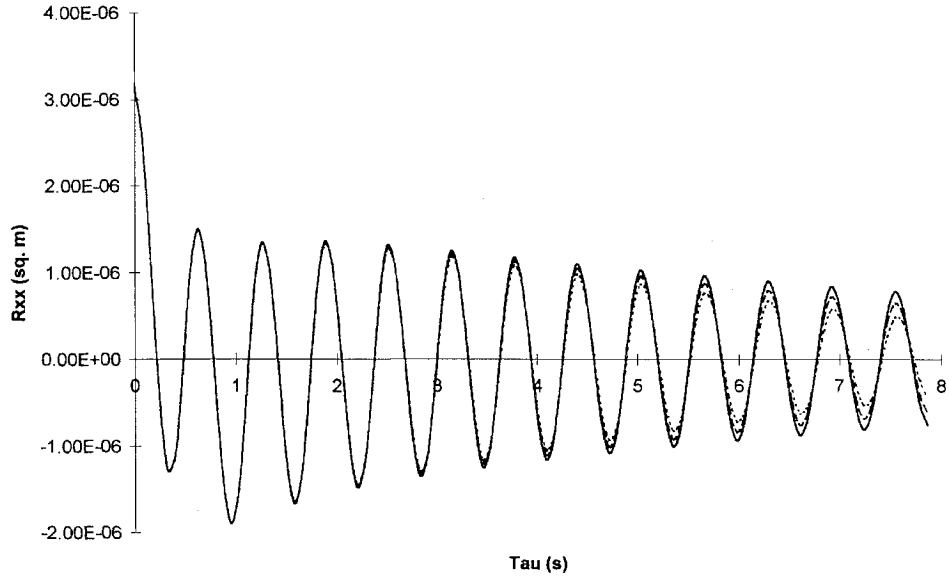

Fig. 5 Autocorrelation of deflection obtained by random matrix analysis for damping ratio $\zeta = 0.01$, stiffness coefficient of variation $COV = 0.01$, and various stiffness correlation parameters: ——, $b = 1$; – – –, $b = 10$; and - - - -, $b = 100$.

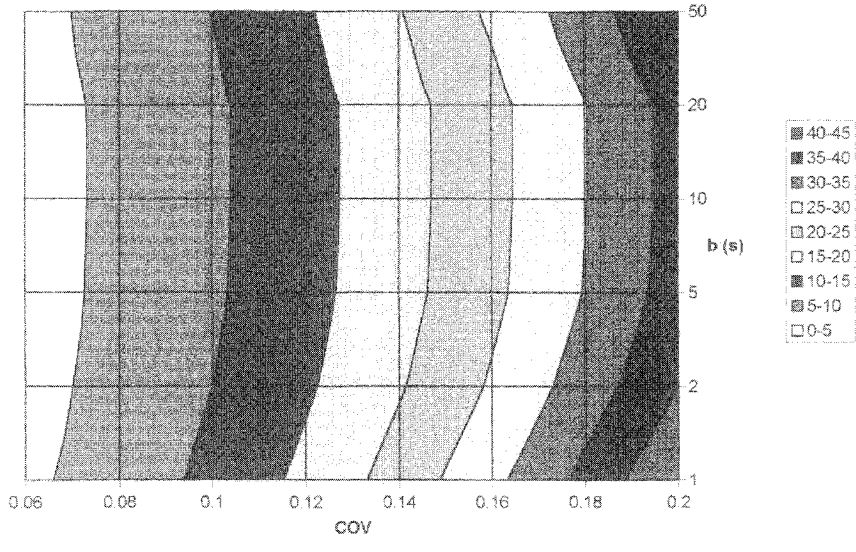

Fig. 6 Percent increase in variance of deflection from deterministic property case for $\zeta = 0.01$ with various coefficients of variation COV and correlation parameters b.

frequency domain (1000 samples), and by a Runge–Kutta Monte Carlo solution in the time domain (1000 samples). The frequency Monte Carlo results agree very well with the direct integration results, implying that enough samples were taken. The Runge–Kutta results agree well for small time shifts τ but show a difference in oscillation frequency and thus become inaccurate for large τ. The random matrix autocorrelation agrees very well for small to moderate τ but experiences a phase difference and increasing amplitudes at larger τ; this is a consequence of the technique's stochastic convergence problem. The random matrix results diverge at a smaller τ for increasing COV, but the deflection variance, which is the autocorrelation at $\tau = 0$, is always reliable.

The second set of comparison results are for a system with a mean damping ratio $\zeta = 0.01$, stiffness correlation parameter $b_k = 1$, and coefficients of variation $COV = 0.08, 0.10$, and 0.12, and results are shown in Fig. 2. Because the correlation parameter is finite, direct integration will not work and a frequency domain Monte Carlo approach would take too much memory. Thus, the random matrix results will be compared with those acquired by Monte Carlo Runge–Kutta analysis. As in the previous set of results, the random matrix autocorrelation agrees well with the Monte Carlo results for small to moderate τ and then shows a phase shift and increasing amplitudes.

Table 4 Time required (hour:minute:second) for various techniques to analyze the problems in this work on an IBM compatible 486DX2/66 desktop computer

Method	$b_k = \infty$	b_k finite
Monte Carlo (Runge–Kutta)	28:56:00	28:56:00
Monte Carlo (frequency)	00:01:20	N/A
Random matrix	00:00:03	00:01:20

The point at which the random matrix results become inaccurate decreases with increasing COV, but those points occur at a larger τ in this case than they did in the previous case. The phase shift and the divergence of the random matrix autocorrelation indicate when the random matrix results become suspect. However, the random matrix results for the variance of the deflection are trustworthy.

Finally, the computation times required for the various approaches to run a single case are shown in Table 4. Since the direct integration results were acquired through commercial software, this technique is not included in Table 4. The Monte Carlo approaches used 1000 samples each, and the computation time for these techniques is linearly dependent on the number of samples. Thus, a case with 500

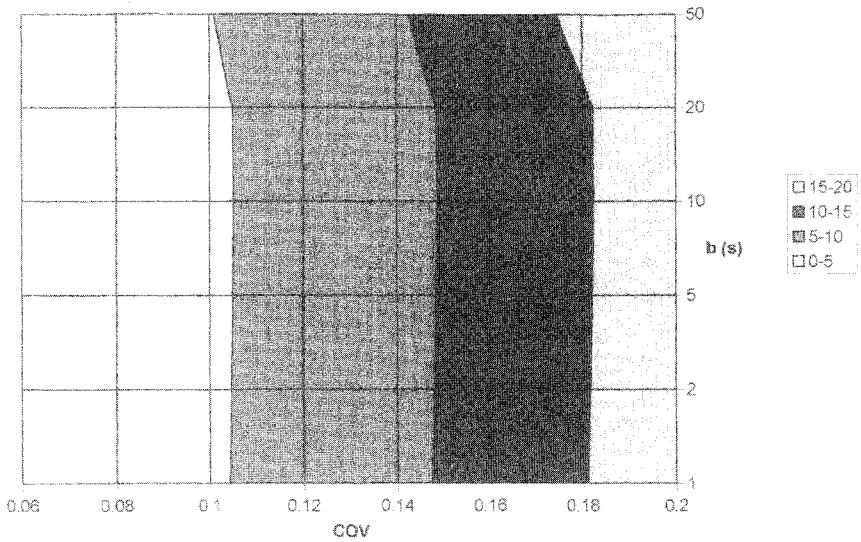

Fig. 7 Percent increase in variance of deflection from deterministic property case for $\zeta = 0.10$ with various coefficients of variation *COV* and correlation parameters *b*.

samples would roughly halve the required computation time. Note that for the Monte Carlo approaches to have comparable computation time with the random matrix approach, the frequency domain Monte Carlo analysis would use only 40 samples; the time-domain analysis could only use one sample. Thus, the random matrix technique is significantly more accurate (for comparable computation time) than the Monte Carlo approaches, while requiring minimal storage.

Effects of Various Parameters on Response

It is helpful at this point to investigate the effects of damping ratio, stiffness *COV*, and stiffness correlation parameter on the autocorrelation of the deflection. All results in this section will be obtained by the random matrix approach, using the same common parameters as the previous results. Note that the response autocorrelation can be roughly defined by two parameters: the value at $\tau = 0$, also called the variance, and the region in τ in which the autocorrelation is significant, also called the correlation time. Formal definitions exist for correlation time but do not apply to the deflection processes found in this work. Therefore, correlation times will only be examined qualitatively; if an autocorrelation decays faster than another, it has a smaller correlation time.

Figure 3 illustrates the effect of damping ratio on the deflection autocorrelation for a system with deterministic stiffness ($COV_k = 0$). Both the deflection variance and the correlation time are higher for lower damping ratios. Note that the results are completely accurate, since there is no property randomness. The effect of stiffness *COV* is shown in Fig. 4 for a system with damping ratio $\zeta = 0.01$ and stiffness correlation parameter $b_k = 1$. Higher *COV* produces a larger variance but a smaller correlation time. Also, the phase shift and divergence of the random matrix results are visible for the cases $COV = 0.16$ and 0.20. At $COV = 0.20$ the Neumann expansion begins to lose validity, and so the results even for small τ may be inaccurate. Finally, Fig. 5 shows the effects of b_k on the deflection autocorrelation for $\zeta = 0.01$ and $COV = 0.01$: the variance is unaffected, but the correlation time decreases with increasing b_k. Figures 3–5 show that the damping ratio is the primary influence on the system behavior (as would be expected), followed by the stiffness coefficient of variation and correlation parameter, in that order.

A more quantitative measure of the effects of random stiffness on the deflection response is found in Figs. 6 and 7 for systems with damping ratios $\zeta = 0.01$ and 0.10, respectively. Figures 6 and 7 show the percent increase in deflection variance from that obtained for deterministic stiffness ($COV_k = 0$) for various stiffness coefficients of variation and correlation parameters. For example, in a system with damping ratio $\zeta = 0.01$, if the stiffness *COV* is less than 6%, the randomness in the stiffness can be ignored, depending on what accuracy is desired for the results. On the other hand, for a system with damping ratio $\zeta = 0.10$, the randomness in the stiffness can be ignored if $COV < 0.10$. Thus, the relative error in variance arising from neglecting random stiffness effects increases with COV_k, decreases with damping ratio, and depends on b_k in a more complicated way.

Conclusions

This work has examined several SDOF systems undergoing FRPV to determine what effects the random properties have on the deflection. To this end, a random matrix method has been developed that is a variation of the perturbation method, but that nevertheless is very fast and requires relatively little storage compared with other computational techniques. Also, the deflection autocorrelation results found by the random matrix technique are accurate for small to moderate time shift τ, and it is relatively easy to see at what point the results become suspect.

Various systems with different damping ratios, stiffness coefficients of variation, and stiffness correlation parameters were analyzed by the aforementioned random matrix technique. Comparisons were made between results acquired by various techniques, showing the strengths and limitations of the random matrix approach. Then, effects of each parameter on the deflection variance and correlation time were examined. Finally, contour plots illustrating the error produced in ignoring random property effects are shown. Plots similar to these for other systems would show engineers under which circumstances random property effects may be safely ignored, saving considerable computation time and money without sacrificing accuracy.

It is possible to extend the random matrix approach to multiple-degree-of-freedom systems that are subjected to either static or dynamic loading. However, it should be mentioned that the current methodology is not compatible with existing deterministic finite element codes. The central idea behind the random matrix approach is the construction of a random matrix-vector equation, where the statistics of the deflection depend upon the statistics of the property matrix elements and the load vector. It would be difficult to use portions of a deterministic finite element program to obtain these needed statistics; it would be better to develop a statistical finite element code that forms a random matrix equation internally and incorporates the random matrix method to obtain the desired output statistics.

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